On Continuous Phase Frequency Shift Keying

1 Continuous Phase BFSK Using a VCO

Binary FSK uses two waveforms to transmit binary information:

\[
A \cos \left( \left( \omega_0 - \frac{\Delta \omega}{2} \right) t + \theta_0 \right) \quad \text{to transmit a 0}
\]

\[
A \cos \left( \left( \omega_0 + \frac{\Delta \omega}{2} \right) t + \theta_1 \right) \quad \text{to transmit a 1}
\]

In order for the phase of the modulated waveform to be continuous on the symbol transition boundaries, the modulator must “remember” what the phase of the modulated carrier was during the previous symbol interval. This requires us to express the modulated waveform not only in terms of the current symbol, but also in terms of the previous symbols. You’ll see that we have to modify our notation a bit to account for this memory. In this section we explore how to use a VCO to generate these signals.

Consider the continuous phase BFSK modulator illustrated in Figure 1. The input to the look-up table (LUT) is a binary sequence consisting of ones and zeros. The output of the look-up table is a zeros of real values

\[ \ldots a(-2), a(-1), a(0), a(1), a(2), \ldots \]

Figure 1: Continuous phase frequency shift keying: the modulator.
where the $k$-th LUT output is
\[ a(k) \in \{-1, +1\}. \] (4)

The D/A converter produces a bipolar square wave whose amplitude during the interval $kT_s \leq t \leq (k+1)T_s$ is $a(k)$. The D/A output can be expressed as
\[ x(t) = \sum_{k=-\infty}^{\infty} a(k)p(t - kT_s) \] (5)

where
\[ p(t) = \begin{cases} 1 & 0 \leq t \leq T_s \\ 0 & \text{otherwise} \end{cases}. \] (6)

The output of the VCO $y(t)$ is related to the input of the VCO $x(t)$ by
\[ y(t) = A \cos \left( \omega_0 t + \omega_d \int_{-\infty}^{t} x(\lambda)d\lambda \right). \] (7)

The amplitude $A$ is usually $\sqrt{2/T_s}$ to give unit energy over a symbol time.

To see how this all fits together, let $a(0)$ be the LUT output during the interval $0 \leq t \leq T_s$ the LUT output is $a(0)$. Let’s take a close look at the phase term in (7) to see how (7) can be expressed in a way that makes more sense:
\[ \omega_d \int_{-\infty}^{t} x(\lambda)d\lambda = \omega_d \int_{0}^{t} a(0)d\lambda + \omega_d \sum_{k=-\infty}^{-1} \int_{kT_s}^{(k+1)T_s} a(k)d\lambda \] (8)
\[ = \omega_d a(0)t + \omega_d T_s \sum_{k=-\infty}^{-1} a(k) \] (9)
\[ = \omega_d a(0)t + \theta \] (10)

Applying this result we see that the VCO output may be expressed as
\[ y(t) = A \cos \left( \omega_0 t + \omega_d a(0)t + \theta \right) \] (11)

for $0 \leq t \leq T_s$. Now suppose $a(0) = -1$. Then the VCO output is
\[ y(t) = A \cos \left( \omega_0 t - \omega_d t + \theta_0 \right) \quad 0 \leq t \leq T_s \] (12)
\[ = A \cos \left( (\omega_0 - \omega_d) t + \theta_0 \right) \quad 0 \leq t \leq T_s \] (13)
which is of the form (1) when \( \omega_d = \frac{\Delta \omega}{2} \). Likewise, when \( a(0) = +1 \), the VCO output is

\[
y(t) = A \cos \left( \omega_0 t + \omega_d t + \theta_0 \right) \quad 0 \leq t \leq T_s
\]

(14)

\[
y(t) = A \cos \left( (\omega_0 + \omega_d) t + \theta_0 \right) \quad 0 \leq t \leq T_s
\]

(15)

which is of the form (2) when \( \omega_d = \frac{\Delta \omega}{2} \).

In general let \( a(k) \) be the LUT output during the interval \( kT_s \leq t \leq (k + 1)T_s \). Then the VCO input may be expressed as

\[
x(t) = a(k)p(t - kT_s) + \sum_{i=-\infty}^{k-1} a(i)p(t - iT_s)
\]

(16)

where \( p(t) \) is given by (6). The corresponding phase term is

\[
\frac{\Delta \omega}{2} \int_{-\infty}^{t} x(\lambda) d\lambda = \frac{\Delta \omega}{2} \int_{kT_s}^{t} a(k) d\lambda + \frac{\Delta \omega T_s}{2} \sum_{i=-\infty}^{k-1} a(i)
\]

(17)

\[
= \frac{\Delta \omega}{2} a(k)(t - kT_s) + \frac{\Delta \omega T_s}{2} \sum_{i=-\infty}^{k-1} a(i)
\]

(18)

\[
= \frac{\Delta \omega}{2} a(k)t - \frac{\Delta \omega}{2} a(k)kT_s + \frac{\Delta \omega T_s}{2} \sum_{i=-\infty}^{k-1} a(i)
\]

constant=\( \theta_k \)

(19)

\[
= \frac{\Delta \omega}{2} a(k)t + \theta_k.
\]

(20)

The VCO output is

\[
y(t) = A \cos \left( \omega_0 t + a(k) \frac{\Delta \omega}{2} t + \theta_k \right) \quad kT_s \leq t \leq (k + 1)T_s
\]

(21)

\[
y(t) = A \cos \left( (\omega_0 + a(k) \frac{\Delta \omega}{2}) t + \theta_k \right) \quad kT_s \leq t \leq (k + 1)T_s.
\]

(22)

The \( k \)-th data bit selects the \( k \)-th look-up table value \( a(k) \) which is either \( \pm 1 \). The value of \( a(k) \) determines whether the frequency shift is up \( (a(k) = +1) \) or down \( (a(k) = -1) \) during the interval \( kT_s \leq t \leq (k + 1)T_s \). The term \( \theta_k \) represents a phase term that produces a continuous phase transition from symbol interval to symbol interval.

Sometimes the frequency shift is normalized to the symbol rate using the \textit{digital modulation index}:

\[
h = \frac{\Delta \omega}{2\pi} T_s.
\]

(23)
In terms of \( h \), the VCO output is

\[
y(t) = A \cos \left( \left( \omega_0 + a(k) \frac{\pi h}{T_s} \right) t + \theta_k \right)
\]

(24)

where the phase term is

\[
\theta_k = \pi h k a(k) + \pi h \sum_{i=-\infty}^{k-1} a(i)
\]

(25)

2 Orthogonality and the Correlation Function

The frequency shift (\( \Delta \omega \) or \( h \)) is often chosen to make the two waveforms (1) and (2) orthogonal. To compute the values of \( \Delta \omega \) or \( h \) needed to accomplish this, we use the correlation function \( R(\Delta \omega) \) defined as follows:

\[
R(\Delta \omega) = \int_0^{T_s} \sqrt{2 \frac{T_s}{T_s}} \cos \left( \left( \omega_0 - \frac{\Delta \omega}{2} \right) t + \theta_0 \right) \sqrt{2 \frac{T_s}{T_s}} \cos \left( \left( \omega_0 + \frac{\Delta \omega}{2} \right) t + \theta_0 \right) dt
\]

(26)

\[
= \frac{\sin(\Delta \omega T_s)}{\Delta \omega T_s}
\]

(27)

which is the familiar sinc() function. We use this to determine orthogonality by noticing that (26) is the definition we have been using for orthogonality. This means that for those values of \( \Delta \omega \) for which \( R(\Delta \omega) \) is zero, the two waveforms are orthogonal over a symbol period. Equation (27) is plotted in Figure 2 (a). Here we see that \( R(\Delta \omega) = 0 \) for \( \Delta \omega T_s = k\pi \) for \( k = \pm1, \pm2, \ldots \). This means orthogonality is achieved for

\[
\Delta \omega = \frac{\pi}{T_s}, \frac{2\pi}{T_s}, \frac{3\pi}{T_s}, \ldots
\]

(28)

or, using

\[
\Delta f = \frac{\Delta \omega}{2\pi}
\]

(29)

the required frequency shift in Hz may be expressed as

\[
\Delta f = \frac{1}{2T_s}, \frac{2}{2T_s}, \frac{3}{2T_s}, \ldots
\]

(30)

This means that all frequency shifts that are multiples of half the symbol rate produce a set of orthogonal signals. The minimum frequency shift that produces orthogonal signals is

\[
(\Delta f)_{\text{min}} = \frac{1}{2T_s}
\]

(31)
which is half the bit rate. BFSK using the minimum frequency shift to produce orthogonal signals is called minimum shift keying or MSK.

Using (23), the correlation function can be expressed in terms of the digital modulation index:

\[ R(h) = \frac{\sin(2\pi h)}{2\pi h}. \] (32)

Equation (32) is plotted in Figure 2 (b). The values of \( h \) which produce orthogonal signals is

\[ h = \frac{1}{2}, 1, \frac{3}{2}, \ldots \] (33)

MSK corresponds to FSK with \( h = 0.5 \).

3 Orthogonal Signaling using Frequency Shift Keying

With proper selection of the frequency shift (or digital modulation index), Equations (1) and (2) can be thought of as orthogonal basis functions that span a signal space. Normalizing the energy
Figure 3: The modulator for binary orthogonal signaling from the basis function point of view.

we get

\[
\phi_0(t) = \begin{cases} 
\sqrt{\frac{2}{T_s}} \cos \left( \left( \omega_0 - \frac{\pi}{T_s} \right) t \right) & 0 \leq t \leq T_s \\
0 & \text{otherwise}
\end{cases}
\]

\[
\phi_1(t) = \begin{cases} 
\sqrt{\frac{2}{T_s}} \cos \left( \left( \omega_0 + \frac{\pi}{T_s} \right) t \right) & 0 \leq t \leq T_s \\
0 & \text{otherwise}
\end{cases}
\]

where the digital modulation index \( h \) is a multiple of 1/2. The corresponding constellation and modulator are shown in Figure 3. In this case, a zero is assigned to \( \phi_0(t) \) and a one is assigned to \( \phi_1(t) \). Note that the entries in the look-up tables are such that when \( a_0 = 1, a_1 = 0 \) and when \( a_0 = 0, a_1 = 1 \). As we have seen, the structure requiring two look-up tables and two multipliers is equivalent to the VCO system shown at the bottom. It should be noted that with the proper phasing of the basis functions, this modulator is identical to the VCO-based modulator of Figure 1.

Even though the VCO system is used in practice, it still helps to think of the modulator in terms of the two basis functions. Thinking of the modulator in this way suggest a the form for the detector shown in Figure 4. Here the detector projects the received signals on to the two basis
functions to produce an estimate \( \hat{a} \) of the coefficients \( a_0 \) and \( a_1 \). The function used by the decision block is illustrated by the decision regions also included in Figure 4. Since the decision region boundary is the line \( \hat{a}_0 = \hat{a}_1 \), the decision regions partition the signal space into two regions: one for which \( \hat{a}_0 > \hat{a}_1 \) and the other for which \( \hat{a}_0 < \hat{a}_1 \). This is a particularly easy decision rule to implement and a couple of alternate methods are illustrated in Figure 5.

The biggest challenge with the detector structure of Figure 4 is producing phase-coherent replicas of the basis functions. This is a non-trivial function that is very hard to do. One way to avoid this problem is to use a detector structure that does not require a phase coherent reference. There are two basic types: the Foster-Sealey detector and the square-law detector.

The Foster-Sealey detector is illustrated in Figure 6. This detector consists of two band-pass filters followed by envelope detectors. One of the band-pass filters is tuned to \( \omega_0 - \frac{\pi f_s}{T_s} \) and the other is tuned to \( \omega_0 + \frac{\pi f_s}{T_s} \). The envelope detector measures the amplitude of the band-pass signal at the output of the band-pass filter. The decision block simply chooses the symbol corresponding to the largest envelope.

An example of how this works is illustrated in Figure 7. Suppose the transmitted signal is 
\[
s(t) = \sqrt{\frac{2}{T_s}} \cos \left( \left( \omega_0 + \frac{\pi f_s}{T_s} \right) t \right).
\]
Ignoring noise for the time being, the output of the band-pass filter tuned to \( \omega_0 + \frac{\pi f_s}{T_s} \) is \( s(t) \) as shown. The output of the bandpass filter tuned to \( \omega_0 - \frac{\pi f_s}{T_s} \)
Figure 5: Two options for the decision block in binary orthogonal signaling.

Figure 6: The Foster-Sealey Detector.
Figure 7: Example of how the Foster-Sealy detector works. Assume a sinusoid at $\omega_0 + \frac{\pi h}{T_s}$ is received. The output of the bandpass filter tuned to $\omega_0 + \frac{\pi h}{T_s}$ is greater than the output of the bandpass filter tuned to $\omega_0 - \frac{\pi h}{T_s}$. As a result, the envelope $X$ is greater than the envelope $Y$ and the correct decision is made.

will be some small amplitude sinusoid at $\omega_0 - \frac{\pi h}{T_s}$ representing the Fourier components of the received signal at $\omega_0 - \frac{\pi h}{T_s}$. Note that if $h$ is chosen so that the two possible transmitted signals are orthogonal, then there is no Fourier component of $s(t)$ at $\omega_0 - \frac{\pi h}{T_s}$. We’ll include some small component here (it’s due to noise) just to illustrate how the decision rule works. The outputs of the envelope detectors are the constant amplitudes of these two sinusoids as shown. The decision rule is based on the sign of $Y - X$ which is the same as choosing 1 if $Y > X$ and 0 otherwise.

Another form of non-coherent detection for binary FSK is illustrated in Figure 8. This detector uses a noncoherent projection of the received signal on to the two basis functions by using a quadrature mixers at each of the two possible frequencies. The outputs are integrated over a symbol time and squared. The quadrature results are summed and passed to the decision block which chooses the symbol associated with the largest output.
Figure 8: Another non-coherent detector for BFSK. This detector uses quadrature mixers at each of the two possible frequencies to compute a non-coherent projection on to the signal space. This projection is non-coherent since no sign information is preserved. The decision is based on the largest non-coherent projection.
To see how this works, let

\[ s_0(t) = \begin{cases} \sqrt{\frac{2E}{T_s}} \cos \left( \left( \omega_0 - \frac{n h}{T_s} \right) t + \theta_0 \right) & 0 \leq t \leq T_s \\ 0 & \text{otherwise} \end{cases} \quad (36) \]

\[ s_1(t) = \begin{cases} \sqrt{\frac{2E}{T_s}} \cos \left( \left( \omega_0 + \frac{n h}{T_s} \right) t + \theta_1 \right) & 0 \leq t \leq T_s \\ 0 & \text{otherwise} \end{cases} \quad (37) \]

be the two possible transmitted signals. Now, suppose the received signal is

\[ r(t) = \sqrt{\frac{2E}{T_s}} \cos \left( \left( \omega_0 + \frac{n h}{T_s} \right) t + \theta_1 \right). \]

Now let's compute the sampled outputs of the four integrators:

\[ y_1 = \int_0^{T_s} \sqrt{\frac{2E}{T_s}} \cos \left( \left( \omega_0 + \frac{n h}{T_s} \right) t + \theta_1 \right) \sqrt{\frac{2E}{T_s}} \cos \left( \left( \omega_0 - \frac{n h}{T_s} \right) t \right) \, dt \]

\[ = \sqrt{\frac{2E}{T_s}} \frac{2}{T_s} \int_0^{T_s} \cos \left( \left( \omega_0 + \frac{n h}{T_s} \right) t + \theta_1 \right) \cos \left( \left( \omega_0 - \frac{n h}{T_s} \right) t \right) \, dt \]

\[ = 0 \quad (38) \]

\[ y_2 = -\int_0^{T_s} \sqrt{\frac{2E}{T_s}} \cos \left( \left( \omega_0 + \frac{n h}{T_s} \right) t + \theta_1 \right) \sqrt{\frac{2E}{T_s}} \sin \left( \left( \omega_0 - \frac{n h}{T_s} \right) t \right) \, dt \]

\[ = -\sqrt{\frac{2E}{T_s}} \frac{2}{T_s} \int_0^{T_s} \cos \left( \left( \omega_0 + \frac{n h}{T_s} \right) t + \theta_1 \right) \sin \left( \left( \omega_0 - \frac{n h}{T_s} \right) t - \pi/2 \right) \, dt \]

\[ = 0 \quad (39) \]

\[ x_1 = \int_0^{T_s} \sqrt{\frac{2E}{T_s}} \cos \left( \left( \omega_0 + \frac{n h}{T_s} \right) t + \theta_1 \right) \sqrt{\frac{2E}{T_s}} \cos \left( \left( \omega_0 + \frac{n h}{T_s} \right) t \right) \, dt \]

\[ = \sqrt{\frac{2E}{T_s}} \frac{1}{T_s} \int_0^{T_s} \cos \theta_1 \, dt \]

\[ = \sqrt{\frac{2E}{T_s}} \cos \theta_1 \quad (40) \]

\[ x_2 = -\int_0^{T_s} \sqrt{\frac{2E}{T_s}} \cos \left( \left( \omega_0 + \frac{n h}{T_s} \right) t + \theta_1 \right) \sqrt{\frac{2E}{T_s}} \sin \left( \left( \omega_0 + \frac{n h}{T_s} \right) t \right) \, dt \]

\[ = \sqrt{\frac{2E}{T_s}} \frac{1}{T_s} \int_0^{T_s} \sin \theta_1 \, dt \]

\[ = \sqrt{\frac{2E}{T_s}} \sin \theta_1 \quad (41) \]
Thus

\[
X = x_1^2 + x_2^2 \\
= E 
\]

\[
Y = y_1^2 + y_2^2 \\
= 0 
\]

(42) (43)

and the correct decision is made.

The key to proper performance is the orthogonality of the two possible transmitted signals. The orthogonality is achieved by choosing \( h \) to satisfy the orthogonality condition regardless of phase. The values of \( h \) that satisfy this are

\[
h = 1, 2, \ldots . 
\]

(44)

The main point is, the minimum frequency shift to obtain orthogonality when using non-coherent detection is twice the minimum frequency shift when using coherent detection. In this way, we can trade bandwidth for detector complexity. The coherent detector is more complex but allows signals with a lower bandwidth (since the minimum frequency is smaller).

4 Generalizations

The general form of continuous phase frequency shift keying is given by

\[
y(t) = A \cos \left( \omega_0 t + \omega_d \int_{-\infty}^{t} x(\lambda) d\lambda \right) 
\]

(45)

where

\[
x(t) = \sum_{k=-\infty}^{\infty} a(k) p(t - kT_s). 
\]

(46)

In the previous, the pulse shape \( p(t) \) was the NRZ pulse shape given by (6). But, as in the case of M-QAM, the pulse shape can really be anything that makes sense. A commonly used pulse shape for continuous phase FSK is the Gaussian pulse shape. The Gaussian pulse shape and its Fourier transform are given by

\[
p(t) = \frac{\sqrt{\pi}}{\alpha} \exp \left\{ -\frac{\pi^2}{\alpha^2 t^2} \right\} 
\]

(47)

\[
P(f) = \exp \left\{ -\alpha^2 f^2 \right\} . 
\]

(48)
The parameter $\alpha$ controls the 3-dB bandwidth $B$ by the relation

$$B = \frac{\sqrt{\frac{1}{2} \ln 2}}{\alpha}.$$  \hspace{1cm} (49)

Even though the Gaussian pulse shape has infinite support in both time and frequency, the decay in frequency is rapid thereby giving it nice spectral properties using practical measures of bandwidth. In real systems, $p(t)$ must be truncated which produces sidelobes which are controlled through the truncation length. Note that the Gaussian pulse shape does not satisfy the Nyquist Theorem and causes intersymbol interference (ISI). The degree of ISI is controlled through the parameter $\alpha$: smaller values of $\alpha$ produce less ISI, but increase the 3-dB bandwidth. So there is an ISI/bandwidth trade-off that must be considered when using this pulse shape.

This pulse shape, coupled with $h = 0.5$ produces a version of minimum shift keying called Gaussian Minimum Shift Keying or GMSK. GMSK using $BT_s = 0.3$ is the modulation used for GSM$^1$, which is the second general digital cellular phone system used in Europe.

The Bluetooth standard also uses continuous phase frequency shift keying. The modulation index is specified in the range $0.28 \leq h \leq 0.35$ and the pulse shape is Gaussian with $BT_s = 0.5$.

5 Discrete-Time Implementations

The discrete-time counterpart of continuous phase FSK is obtained by assuming a sampling rate $F_s = 1/t_s$ high enough so that integration in continuous-time is well approximated by summation in discrete-time. The sampled version of the signal is obtained by replacing $t$ with $nt_s$ and the integral with a summation:

$$y(t) = A \cos \left( \omega_0t + \omega_d \int_{-\infty}^{t} x(\lambda)d\lambda \right)$$

$$y(nt_s) = A \cos \left( \omega_0nt_s + \omega_d t_s \sum_{i=-\infty}^{n-1} x(it_s) \right)$$  \hspace{1cm} (51)

Using $\Omega_0 = \omega_0 t_s$ and $\Omega_d = \omega_d t_s$ this becomes

$$y(nt_s) = A \cos \left( \Omega_0 n + \Omega_d \sum_{i=-\infty}^{n-1} x(it_s) \right).$$  \hspace{1cm} (52)

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$^1$GSM stands for the French phrase Groupe Spécial Mobile. Recently, GSM changed its name to Global System for Mobile Communications for marketing reasons.
As before, we associate the sequence $y[n]$ with the sampled signal $y(nt_s)$ and the sequence $x[i]$ with the sampled signal $x(it_s)$:

$$y[n] = A \cos \left( \Omega_0 n + \Omega_d \sum_{i=-\infty}^{n-1} x[i] \right). \quad (53)$$

In terms of frequency shift, the relationship

$$\omega_d = \frac{\Delta \omega}{2}$$

implies the relationship

$$\Omega_d = \frac{\Delta \Omega}{2} \quad (\Delta \Omega = \Delta \omega t_s)$$

so that the sequence may be expressed as

$$y[n] = A \cos \left( \Omega_0 n + \frac{\Delta \Omega}{2} \sum_{i=-\infty}^{n-1} x[i] \right). \quad (54)$$

Using $N$ to denote the ratio of sample rate to symbol rate, the digital modulation index may be expressed as

$$h = \frac{\Delta \omega}{2\pi} T_s = \frac{\Delta \omega}{2\pi} N t_s = \frac{\Delta \Omega}{2\pi} N$$

so that the sequence may also be expressed in terms of the digital modulation index:

$$y[n] = A \cos \left( \Omega_0 n + \frac{\pi h}{N} \sum_{i=-\infty}^{n-1} x[i] \right). \quad (56)$$

The sequence $x[i]$ is a sampled version of the input signal $x(t)$. Starting with

$$x(t) = \sum_{k=-\infty}^{\infty} a(k) p(t - kT_s), \quad (57)$$

we obtain the sampled version by replacing $t$ with $it_s$:

$$x(it_s) = \sum_{k=-\infty}^{\infty} a(k) p(it_s - kT_s). \quad (58)$$

Again using $N = T_s/t_s$ we see that

$$p(it_s - kT_s) = p(it_s - kN t_s)$$

$$= p((i - kN)t_s) \quad (60)$$

$$\rightarrow p[i - kN]. \quad (61)$$
Thus, the sequence $x[i]$ associated with samples of $x(t)$ may be expressed as

$$x[i] = \sum_{k=-\infty}^{\infty} a(k) p[i - kN]$$

(62)

A discrete-time version of the continuous phase FSK modulator uses a discrete-time VCO as shown in Figure 9.

Binary orthogonal signaling results when the modulation index is chosen to produce a frequency shift such that the two signals are orthogonal over a symbol period. Using the discrete-time signals and the discrete-time equivalent of the correlation function, we see that the conditions for $h$ are the same as in the continuous-time case. (This shouldn’t be surprising since the discrete-time system is equivalent to the continuous-time system.) To see that this is so, let the two possible discrete-time signals be

$$s_0[n] = \begin{cases} \sqrt{\frac{2}{N}} \cos \left( \left( \Omega_0 - \frac{\pi h}{N} \right) n \right) & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$$

(63)

$$s_1[n] = \begin{cases} \sqrt{\frac{2}{N}} \cos \left( \left( \Omega_0 + \frac{\pi h}{N} \right) n \right) & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$$

(64)
and compute the correlation function

\[ R(h) = \sum_{n=0}^{N-1} s_0[n] s_1[n] \]  
\[ = \frac{2}{N} \sum_{n=0}^{N-1} \cos \left( \left( \Omega_0 - \frac{\pi h}{N} \right) n \right) \cos \left( \left( \Omega_0 + \frac{\pi h}{N} \right) n \right) \]  
\[ = \frac{\sin \left( 2\pi h - \frac{\pi h}{N} \right) - \sin \left( \frac{\pi h}{N} \right)}{2N \sin \left( \frac{\pi h}{N} \right)} \]  

(65)  
(66)  
(67)

For sampling rates high enough so that continuous-time integration is well approximated by summation in discrete-time, \( N \) is large enough so that the correlation function is well approximated by

\[ R(h) = \frac{\sin \left( 2\pi h \right)}{2\pi h} \]  

(68)

which is identical to (32). The discrete-time equivalent of the modulator for binary orthogonal signaling is shown in Figure 10.

The detectors are discrete-time counterparts of the continuous-time systems as illustrated in the Figures 11 through 13.
\[ y(t) = a_o(k)\phi_0(t - kT_s) + a_i(k)\phi_i(t - kT_s) \]

\[ kT_s \leq t \leq (k + 1)T_s \]

Figure 10: Discrete-time equivalent modulator for binary orthogonal signaling.
Figure 11: Discrete-time equivalent detector for binary orthogonal signaling.

Figure 12: Discrete-time equivalent Foster-Sealy detector for binary orthogonal signaling.
Figure 13: Discrete-time equivalent of the square-law detector for binary orthogonal signaling.